A COMMENT ON THE DEFINITION OF RELATIVE PRESSURE

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Abstract. We show that two natural definitions of the relative pressure function for a locally constant potential function and a factor map from a shift of finite type coincide almost everywhere with respect to every invariant measure. With a suitable extension of one of the definitions, the same holds true for any continuous potential function.

The introduction to the paper [3] included, for factor maps between subshifts and the identically zero potential, a "finite-range" definition of the relative pressure function that is different from the standard one [1, 6], which involves complete bisequences. While this variation in the definition had no bearing on the results of that paper, it does seem useful to clarify the extent to which the two definitions differ; in particular, in some situations one definition may be easier to use than the other. We show in this note that for a factor map $\pi : X \to Y$, where X is a shift of finite type and Y is a subshift, the two relative pressure functions can be different, but they coincide almost everywhere with respect to every invariant measure on Y. Therefore, for each ergodic invariant measure ν on Y, the two definitions lead to the same value of the maximal possible relative entropy $h_\mu(X|Y)$ of any invariant measure μ on X over Y. More generally, for any locally constant potential function (one that depends on only finitely many coordinates), the analogously defined two relative pressure functions coincide almost everywhere with respect to every invariant measure on Y. Finally, we show that this statement continues to hold for an arbitrary continuous potential function with a suitably generalized definition of the finite-range relative pressure function.

Several useful ideas for the study of the relative entropy function were developed in [5] in connection with questions about the existence of compensation functions, and we adopt and extend them for our purposes here.

If (X, S) is a topological dynamical system, then $M(X)$ will denote the set of all S-invariant Borel probability measures on X with its weak* topology. Given $x \in X$, let

$$\mu_x = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{S^i x} \in M(X)$$

if it exists, in which case we call x a *generic point*. Denote by $\mathcal{G}(X)$ the set of all generic points in X. If X is a shift space, then for each $n \geq 1$, $\mathcal{B}_n(X)$ denotes the set of n-blocks in X, and $\mathcal{B}(X) = \cup_n \mathcal{B}_n(X)$. Given $b_1 \cdots b_n \in \mathcal{B}_n(X)$, $n \geq 1$, we define

$${}_r[b_1 \cdots b_n]_{r+n-1} = \{x \in X : x_r = b_1, \ldots, x_{r+n-1} = b_n\},$$

and we abbreviate ${}_0[b_1 \cdots b_n]_{n-1}$ by $[b_1 \cdots b_n]$. We denote the shift transformation by σ and the usual metric for a subshift by ρ.

Let $S : X \to X$ and $T : Y \to Y$ be continuous maps of compact metrizable spaces and $\pi : X \to Y$ a factor map. i.e., a continuous surjection with $\pi \circ S = T \circ \pi$. For a given compact subset K of X, for $n \geq 1$ and $\delta > 0$, denote by $\Delta_{n,\delta}(K)$ the set of (n, δ)-separated sets of X contained in K. Let $f \in C(X)$. Fix $\delta > 0$ and $n \geq 1$. For each $y \in Y$, let

$$P_n(\pi, f, \delta)(y) = \sup \left\{ \sum_{x \in E} \exp\left(\sum_{i=0}^{n-1} f(S^i x) \right) \middle| E \in \Delta_{n,\delta}\big(\pi^{-1}\{y\}\big) \right\}.$$

Define $P(\pi, f) : Y \to \mathbb{R}$ by

$$P(\pi, f)(y) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln P_n(\pi, f, \delta)(y).$$

The function $P(\pi, f)$ is called the *relative pressure function* associated with f. It is Borel measurable and T-invariant. For $\nu \in M(Y)$, let $M(\nu) = \pi^{-1}(\nu)$ denote the set of measures in $M(X)$ that project to ν under π. Given $f \in C(X)$, the function $P(\pi, f) : Y \to \mathbb{R}$ satisfies the *relative variational principle* [1]: for each $\nu \in M(Y)$,

$$\int P(\pi, f)\, d\nu = \sup \left\{ h(\mu) + \int f d\mu \middle| \mu \in M(\nu) \right\} - h(\nu).$$

In particular, for a fixed $\nu \in M(Y)$,

$$\sup\{h_\mu(X|Y) : \mu \in M(\nu)\} = \sup\{h(\mu) - h(\nu) : \mu \in M(\nu)\} = \int_Y P(\pi, 0)\, d\nu.$$

Hereinafter, let X be a shift of finite type and Y a subshift, on finite alphabets $\mathcal{A}(X)$ and $\mathcal{A}(Y)$, respectively. Let $\pi : X \to Y$ be a factor map, so that Y is a sofic system. We treat the 2-sided case, the 1-sided case being very similar. Let $f \in \mathcal{C}(X)$ be a locally constant function, i.e. one that depends on only finitely many coordinates $x_{-m} \cdots x_m$. By passing to a higher block representation if necessary, we may assume that π is represented by a one-block map from $\mathcal{B}_1(X)$ to $\mathcal{B}_1(Y)$, which we denote again by π, and that f is a function of the two coordinates $x_0 x_1$.

For $B = b_1 \cdots b_n \in \mathcal{B}_n(X)$, $\pi(B)$ means the n-block $\pi(b_1) \cdots \pi(b_n)$ of Y; given $v \in \mathcal{B}_n(Y)$, $\pi^{-1}(v)$ denotes the set of n-blocks of X that project to v by the block map π. Given $y \in Y$, for each $n \geq 1$, let $D_n(y)$ consist of one point from each nonempty set $\pi^{-1}(y) \cap [x_0 x_1 \cdots x_{n-1}]$. The potential function f determines a block map $F : \mathcal{B}_2(X) \to \mathbb{R}$ by $F(b_0 b_1) = \exp(f(x))$ for any $x \in [b_0 b_1]$. For a block $B = b_1 \cdots b_n \in \mathcal{B}(X)$, put

$$s_f(B) = F(b_1 b_2) F(b_2 b_3) \cdots F(b_{n-1} b_n),$$

and for a block $w \in \mathcal{B}(Y)$, put $S_f(w) = \sum_B s_f(B)$ where the sum is taken over all $B \in \mathcal{B}(X)$ that are mapped to w by π. Then for each $y \in Y$,

$$
\begin{aligned}
P(\pi, f)(y) &= \limsup_{n\to\infty} \frac{1}{n} \ln \left[\sum_{x \in D_n(y)} \exp\Big(\sum_{i=0}^{n-1} f(\sigma^i x) \Big) \right] \\
&= \limsup_{n\to\infty} \frac{1}{n} \ln \left[\sum_{x \in D_n(y)} s_f(x_0 x_1 \cdots x_{n-1}) \right]
\end{aligned} \tag{1}
$$

(see [6, Theorem 4.6]). In particular, for all $y \in Y$,

$$P(\pi, 0)(y) = \limsup_{n\to\infty} \frac{1}{n} \ln \big|D_n(y)\big|$$

(with $f \equiv 0$). Define another Borel-measurable function $\Phi_f : Y \to \mathbb{R}$ by

$$\Phi_f(y) = \limsup_{n\to\infty} \frac{1}{n} \ln S_f(y_0 y_1 \cdots y_{n-1}) \qquad \text{for } y \in Y.$$

It can be shown that $\Phi_f(y) \leq \Phi_f(\sigma y)$ for all $y \in Y$. Also $P(\pi, f)(y) \leq \Phi_f(y)$ for all $y \in Y$. Given $f \in C(X)$, one may have $P(\pi, f)(y) < \Phi_f(y)$ for some $y \in Y$, as seen in the following example (which in [4] and [5] was shown to be a factor map for which there exists no saturated compensation function).

Example 1. Let X, Y be the subshifts of finite type determined by allowing the transitions marked on Figure 1 and the one-block factor code $\pi : X \to Y$ map 1 to 1, and $2, 3, 4, 5$ to 2.

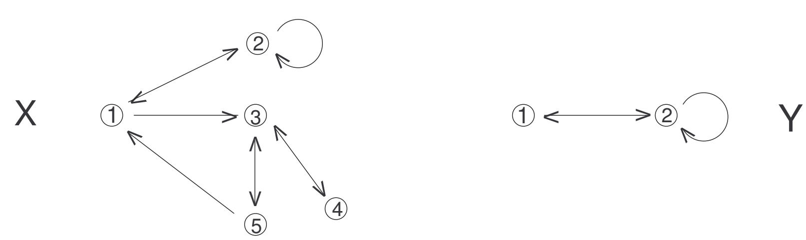

FIGURE 1.

For each $k \geq 1$, let $a_k = 2^k + 1$, and define $y \in Y$ by

$$y = \cdots 222.12^{a_1} 12^{a_2} 12^{a_3} 1 \cdots$$

(so that $y_i = 2$ for all $i < 0$). Whenever $\pi x = y$, then $x_{[0,\infty)} = y_{[0,\infty)} = 12^{a_1} 12^{a_2} 12^{a_3} 1 \cdots$, since each a_k is odd and so $\pi^{-1}(12^{a_k} 1) = \{12^{a_k} 1\}$. Thus $|D_n(y)| = 1$ for all $n \geq 1$ which implies that $P(\pi, 0)(y) = 0$. Meanwhile, fix $k \geq 1$ and set $n_k = 2^{k+1} + 2k - 2$. Then

$$\big|\pi^{-1}[y_0 y_1 \cdots y_{n_k - 1}]\big| = \big|\pi^{-1}[12^{a_1} 12^{a_2} 1 \cdots 12^{a_k}]\big| = \big|\pi^{-1}[12^{a_k}]\big| = 2^{2^{k-1}} + 1.$$

Consider $f \equiv 0 \in C(X)$. Then

$$\Phi_f(y) \geq \limsup_{k\to\infty} \frac{1}{n_k} \ln \left|\pi^{-1}[y_0y_1\cdots y_{n_k-1}]\right|$$
$$= \lim_{k\to\infty} \frac{\ln\left(2^{2^{k-1}}+1\right)}{2^{k+1}+2k-2} = \frac{\ln 2}{4} > 0 = P(\pi,0)(y).$$

Our goal is to prove the following.

Theorem 1. *Let X be an irreducible shift of finite type, Y a subshift, and $\pi : X \to Y$ a factor map. Let $f \in C(X)$ be a function which depends on just 2 coordinates, x_0x_1, of each point $x \in X$. Then for each $\nu \in M(Y)$, we have $P(\pi, f)(y) = \Phi_f(y)$ a.e. $d\nu(y)$, and hence*

$$\int P(\pi, f)\, d\nu = \int \Phi_f\, d\nu.$$

Remark. We may assume that $f \geq 0$. For if f takes some negative values, choose a constant c such that $f + c \geq 0$ and use the equations $P(\pi, f + c) = P(\pi, f) + c, \Phi_{f+c} = \Phi_f + c$. Thus we have $1 \leq F \leq M$ for some constant M.

For notational convenience, set $s = s_f$, $S = S_f$ and $\Phi = \Phi_f$, and let

$$\mathcal{T}(y) = P(\pi, f)(y) \qquad \text{for } y \in Y.$$

The map $\mathcal{R} : M(X) \to \mathbb{R}^+$ defined by $\mathcal{R}(\mu) = h(\mu) - h(\pi\mu)$ is upper semicontinuous in the weak* topology [6, Lemma 2.2]. Using this one can easily prove the following [5].

Lemma 1. *For any $f \in C(X)$ the affine map $\mathcal{L} : M(Y) \to \mathbb{R}$ given by $\mathcal{L}(\nu) = \int \mathcal{T}\, d\nu$ is upper semicontinuous.*

For $q \geq 1$, let $P_q(Y) = \{y \in Y | \sigma^q(y) = y\}$ and $P(Y) = \bigcup_{q\geq 1} P_q(Y)$.

Lemma 2. *Let $y \in P_q(Y)$, $q \geq 1$. Then*

$$\Phi(y) = \lim_{n\to\infty} \frac{1}{nq} \ln S(y_0y_1\cdots y_{nq-1}). \tag{2}$$

Also $\Phi(y) = \mathcal{T}(y)$.

Proof. For a block $w \in \mathcal{B}(Y)$ with $ww \in \mathcal{B}(Y)$, we have $S_f(w^{n+m}) \leq M \cdot S_f(w^n)S_f(w^m)$. Thus $(1/n)\ln S_f(w^n)$ converges as $n \to \infty$ (see [2, p. 240]), and hence

$$\Phi(y) = \lim_{n\to\infty} \frac{1}{nq} \ln S(y_0\cdots y_{nq-1}).$$

To show that $\Phi(y) = \mathcal{T}(y)$, set $w = y_0y_1\cdots y_{q-1} \in \mathcal{B}_q(Y)$, so that

$$y = \cdots w.ww\cdots = \cdots y_{q-1}.y_0y_1\cdots y_{q-1}y_0y_1\cdots.$$

Let $|\pi^{-1}(w)| = l \geq 1$ and define an $l \times l$, 0-1 matrix $A = (A_{uv})$ by $A_{uv} = 1$ if and only if $uv \in \mathcal{B}_{2q}(X)$, where $u, v \in \pi^{-1}(w)$, that is, A is the transition matrix between blocks in the

inverse image of the repeating word w that forms y. Note that some blocks in $\pi^{-1}(w)$ may be preceded or followed only by allowable q-blocks in X that do not map to w. So we let B be the reduction of A obtained by excluding all the zero columns and rows together with their corresponding rows and columns. Then by (2),

$$\Phi(y) = \lim_{n\to\infty} \frac{1}{nq} \ln \Bigg[\sum_{u_i \in \pi^{-1}(w)} s(u_1 \cdots u_n) A_{u_1 u_2} \cdots A_{u_{n-1} u_n} \Bigg].$$

Since B is essential, we have

$$\begin{aligned} \sum_{u_i \in \pi^{-1}(w)} & s(u_1 \cdots u_{n+1} u_{n+2}) A_{u_1 u_2} \cdots A_{u_{n+1} u_{n+2}} \\ &\le \sum_{u_i \in \pi^{-1}(w)} \big[s(u_1 \cdots u_n) B_{u_1 u_2} \cdots B_{u_{n-1} u_n} \big] l^2 M^{2q} \\ &\le l^2 M^{2q} \sum_{x \in D_{nq}(y)} s(x_0 x_1 \cdots x_{nq-1}) \\ &\le l^2 M^{2q} \sum_{u_i \in \pi^{-1}(w)} s(u_1 \cdots u_n) A_{u_1 u_2} \cdots A_{u_{n-1} u_n}. \end{aligned}$$

Now we take logarithms, divide, and take the limit on n to get $\Phi(y) = \mathcal{T}(y)$. □

For the proof of the following result, we refer to [5].

Lemma 3. *Let $y \in \mathcal{G}(Y)$ and let $y^{(s)} \in P_{l_s}(Y)$, $l_s \ge s$, for each $s \ge 1$. If there is $N \ge 1$ such that $y^{(s)}_{[0,l_s-N]} = y_{[0,l_s-N]}$ for all s large enough, then $\mu_{y^{(s)}} \to \mu_y$ as $s \to \infty$.*

Let $\mathcal{A}$ now denote the alphabet of X. Fix $y \in Y$. Given $b, c \in \mathcal{A}$ and $n \ge 1$, let

$$\Gamma_y^n(b,c) = \sum_u S(buc),$$

where the sum is taken over all u's in $\mathcal{B}_{n-1}(X)$ such that $\pi(buc) = y_0 y_1 \cdots y_n$. (If $\Gamma_y^n(b,c) \ge 1$, then $\pi b = y_0$ and $\pi c = y_n$.) Then

$$\sum_{b,c \in \mathcal{A}} \Gamma_y^n(b,c) = S(y_0 y_1 \cdots y_n).$$

It is not difficult to check the following.

Lemma 4. *Let $y \in P_q(Y)$, $q \ge 1$, and $b \in \mathcal{A}$. Then for $k \ge 1$,*

$$\big[\Gamma_y^q(b,b)\big]^k \le \Gamma_y^{qk}(b,b).$$

Lemma 5. *Let $y \in \mathcal{G}(Y)$. Then there is a sequence $\{y^{(s)}\}_{s=1}^{\infty} \subset P(Y)$ such that $\mu_{y^{(s)}} \to \mu_y$ as $s \to \infty$ and $\Phi(y) \le \liminf_{s\to\infty} \Phi(y^{(s)})$.*

Proof. Observe first that there exist a symbol, say a, of $\mathcal{A}(Y)$ and a strictly increasing sequence $\{m_k\}_{k=0}^{\infty} \subset \mathbb{Z}^+$ such that $y_{m_k} = a$ for all $k \geq 0$ and

$$\Phi(y) = \lim_{m_k \to \infty} \frac{1}{m_k} \ln S(y_0 \cdots y_{m_k}). \tag{3}$$

Let $y^* = \sigma^{m_0} y \in Y$ and for each $k \geq 0$ put $n_k = m_k - m_0 \geq 0$. Then $y_i^* = a$ if $i = n_k$ for some $k \geq 0$ ($n_0 = 0$). Let $\mathcal{C} = \pi^{-1}(a)$. For $k \geq 1$, choose $b_k, c_k \in \mathcal{C}$ so that

$$\Gamma_{y^*}^{n_k}(b_k, c_k) = \max_{b,c \in \mathcal{C}} \Gamma_{y^*}^{n_k}(b, c)$$

(so $\pi(b_k) = y_0^*$ and $\pi(c_k) = y_{n_k}^*$). Since $1 \leq F \leq M$, it follows that

$$\frac{1}{M^2} S(y_0^* \cdots y_{n_k}^*) \leq \Gamma_{y^*}^{n_k}(b_k, c_k) \leq S(y_0^* \cdots y_{n_k}^*).$$

Thus by (3),

$$\begin{aligned} \Phi(y) &\leq \liminf_{k \to \infty} \frac{1}{m_k} \ln \left[M^{m_0} \cdot S(y_{m_0} \cdots y_{m_k}) \right] \\ &= \liminf_{k \to \infty} \frac{1}{n_k} \ln S(y_0^* \cdots y_{n_k}^*) = \liminf_{k \to \infty} \frac{1}{n_k} \ln \left[\Gamma_{y^*}^{n_k}(b_k, c_k) \right]. \end{aligned} \tag{4}$$

Notice that there exist $b_*, c_* \in \mathcal{C}$ such that $b_k = b_*$ and $c_k = c_*$ for infinitely many k's, say k_s's, where $k_s \nearrow \infty$ as $s \to \infty$. Since X is irreducible, there is $w \in \mathcal{B}_d(X)$ for some $d \geq m_0$ such that $c_* w b_* \in \mathcal{B}_{d+2}(X)$. Let $a_1 \cdots a_d = \pi(w) \in \mathcal{B}_d(Y)$. Fix $s \geq 1$ and put $l_s = n_{k_s} + 1 + d$. Define $y^{(s)} \in P_{l_s}(Y)$ by

$$y^{(s)} = \cdots a_d \,.\, y_0^* y_1^* \cdots y_{n_{k_s}}^* a_1 \cdots a_d \, y_0^* y_1^* \cdots$$

($y_0^* \ldots y_{n_{k_s}}^*$ is the image under π of a word $w^* = w_0^* \cdots w_{n_{k_s}}^*$ in X with $w_0^* = b_k = b_*$ and $w_{n_{k_s}}^* = c_k = c_*$, so that $\cdots w.w^* w w^* w \cdots$ is a legitimate point $x^{(s)} \in X$, and $y^{(s)} = \pi(x^{(s)})$). Since $y_{[0,l_s-d]}^{(s)} = y_{[0,l_s-d]}^*$ and $l_s \geq m_{k_s} \geq k_s \geq s$, it follows from Lemma 3 that $\mu_{y^{(s)}} \to \mu_{y^*}$ or equivalently $\mu_{y^{(s)}} \to \mu_y$ as $s \to \infty$. To see that $\Phi(y) \leq \liminf_{s \to \infty} \Phi(y^{(s)})$, fix $s \geq 1$. By Lemma 4,

$$\begin{aligned} \Phi(y^{(s)}) &= \lim_{p \to \infty} \frac{1}{p \cdot l_s} \ln S(y_0^{(s)} \cdots y_{p \cdot l_s}^{(s)}) \geq \limsup_{p \to \infty} \frac{1}{p \cdot l_s} \ln \left[\Gamma_{y^{(s)}}^{p \cdot l_s}(b_*, b_*) \right] \\ &\geq \limsup_{p \to \infty} \frac{1}{l_s} \ln \left[\Gamma_{y^{(s)}}^{l_s}(b_*, b_*) \right] = \frac{1}{l_s} \ln \left[\Gamma_{y^{(s)}}^{l_s}(b_*, b_*) \right]. \end{aligned}$$

It is clear that $\Gamma_{y^*}^{n_{k_s}}(b_*, c_*) \leq \Gamma_{y^{(s)}}^{l_s}(b_*, b_*)$. Thus from (4),

$$\begin{aligned} \Phi(y) &\leq \liminf_{s \to \infty} \frac{1}{n_{k_s}} \ln \left[\Gamma_{y^*}^{n_{k_s}}(b_{k_s}, c_{k_s}) \right] = \liminf_{s \to \infty} \frac{1}{n_{k_s}} \ln \left[\Gamma_{y^*}^{n_{k_s}}(b_*, c_*) \right] \\ &\leq \liminf_{s \to \infty} \frac{1}{l_s} \ln \left[\Gamma_{y^{(s)}}^{l_s}(b_*, b_*) \right] \leq \liminf_{s \to \infty} \Phi(y^{(s)}), \end{aligned}$$

which completes the proof. □

Let $E = \{y \in \mathcal{G}(Y) | \int \mathcal{T} d\mu_y = \mathcal{T}(y)\}$. Then $\nu(E) = 1$ for every ergodic invariant measure ν on Y, and hence $\nu(E) = 1$ for every $\nu \in M(Y)$. For $y \in E$, let $\{y^{(s)}\}_{s=1}^{\infty} \subset P(Y)$ be a sequence obtained from Lemma 5 so that $\mu_{y^{(s)}} \to \mu_y$ as $s \to \infty$ and $\Phi(y) \leq \liminf_{s\to\infty} \Phi(y^{(s)})$. By Lemma 1,

$$\limsup_{s\to\infty} \mathcal{T}(y^{(s)}) = \limsup_{s\to\infty} \int \mathcal{T}\, d\mu_{y^{(s)}} \leq \int \mathcal{T}\, d\mu_y = \mathcal{T}(y).$$

It follows from Lemma 2 that $\mathcal{T}(y^{(s)}) = \Phi(y^{(s)})$ for all $s \geq 1$. Thus

$$\Phi(y) \leq \liminf_{s\to\infty} \Phi(y^{(s)}) = \liminf_{s\to\infty} \mathcal{T}(y^{(s)}) \leq \mathcal{T}(y) \leq \Phi(y).$$

Hence $\mathcal{T}(y) = \Phi(y)$ for all $y \in E$. Let $\nu \in M(Y)$. Since $\nu(E) = 1$, we have

$$\int \mathcal{T}\, d\nu = \int_E \mathcal{T}\, d\nu = \int_E \Phi\, d\nu = \int \Phi\, d\nu,$$

which completes the proof of Theorem 1.

To extend Theorem 1 to the case of an arbitrary potential $f \in C(X)$, we need to define a suitable function corresponding to Φ_f. Let $f \in C(X)$. Fix $n \geq 1$. For each $i = 0, 1, \cdots, n-1$, define a block map $F_n^i : \mathcal{B}_n(X) \to \mathbb{R}$ by

$$F_n^i(b_1 \cdots b_n) = \inf_{\sigma^{-i}x \in [b_1 \cdots b_n]} \exp(f(x))$$

(so the infimum is taken over all x in the cylinder set ${}_{-i}[b_1 \cdots b_n]_{n-i-1}$). For each n-block $B \in \mathcal{B}_n(X)$, put

$$s_f(B) = \prod_{i=0}^{n-1} F_n^i(B),$$

and for a block $C \in \mathcal{B}_n(Y)$ put

$$S_f(C) = \sum_{\pi(B)=C} s_f(B).$$

Define $\Psi_f : Y \to \mathbb{R}$ by

$$\Psi_f(y) = \limsup_{n\to\infty} \frac{1}{n} \ln S_f(y_0 y_1 \cdots y_{n-1}) \qquad \text{for } y \in Y.$$

Now, given $n \geq 1$, for each $i = 0, 1, \cdots, n-1$ define $\widetilde{F}_n^i : \mathcal{B}_n(X) \to \mathbb{R}$ by

$$\widetilde{F}_n^i(b_1 \cdots b_n) = \sup_{\sigma^{-i}x \in [b_1 \cdots b_n]} \exp(f(x)).$$

Using $\widetilde{F}_n^i$, we similarly define $\widetilde{s}_f, \widetilde{S}_f$ and $\widetilde{\Psi}_f$ as follows. For each $B \in \mathcal{B}_n(X)$,

$$\widetilde{s}_f(B) = \prod_{i=0}^{n-1} \widetilde{F}_n^i(B),$$

and for $C \in \mathcal{B}_n(Y)$,

$$\widetilde{S}_f(C) = \sum_{\pi(B)=C} \widetilde{s}_f(B).$$

Define $\widetilde{\Psi}_f : Y \to \mathbb{R}$ by

$$\widetilde{\Psi}_f(y) = \limsup_{n\to\infty} \frac{1}{n} \ln \widetilde{S}_f(y_0 y_1 \cdots y_{n-1}) \qquad \text{for } y \in Y.$$

It can be easily shown that if f depends on just 2 coordinates, $x_0 x_1$, of each point $x \in X$, then Ψ_f and $\widetilde{\Psi}_f$ both are equivalent to Φ_f, and hence Theorem 1 would apply. In the current more general situation, we still have the analogous result.

Theorem 2. *Let X be an irreducible shift of finite type, Y a subshift, $\pi : X \to Y$ a factor map, and $f \in C(X)$. For each $\nu \in M(Y)$, we have $P(\pi, f)(y) = \Psi_f(y) = \widetilde{\Psi}_f(y)$ a.e. $d\nu(y)$, and hence*

$$\int P(\pi, f)\, d\nu = \int \Psi_f\, d\nu = \int \widetilde{\Psi}_f\, d\nu.$$

Proof. As before, we may assume that $f \geq 0$. Thus we have a constant $M > 0$ such that $1 \leq \exp(f) \leq M$ and hence $1 \leq F_n^i \leq \widetilde{F}_n^i \leq M$ for all $n \geq 1$ and for $i = 0, 1, \cdots, n-1$. Let $\mathcal{T} = P(\pi, f)$ as before. Set $s = s_f$, $S = S_f$, and $\Psi = \Psi_f$. For $y \in Y$, define

$$\theta(y) = \limsup_{n\to\infty} \frac{1}{n} \ln \Bigg[\sum_{x \in D_n(y)} s(x_0 x_1 \cdots x_{n-1}) \Bigg].$$

We show that $\mathcal{T}(y) = \theta(y)$ for all $y \in Y$. Note first that if $x_0 \cdots x_{n-1} \in \mathcal{B}_n(X)$, $n \geq 1$, then

$$\begin{aligned} s(x_0 \cdots x_{n-1}) &= \prod_{i=0}^{n-1} \inf_{\sigma^{-i} z \in [x_0 \cdots x_{n-1}]} \exp(f(z)) \\ &= \prod_{i=0}^{n-1} \inf_{z \in [x_0 \cdots x_{n-1}]} \exp(f(\sigma^i z)). \end{aligned}$$

Thus from (1), given $y \in Y$,

$$\begin{aligned} \theta(y) &= \limsup_{n\to\infty} \frac{1}{n} \ln \Bigg[\sum_{x \in D_n(y)} \prod_{i=0}^{n-1} \inf_{z \in [x_0 \cdots x_{n-1}]} \exp(f(\sigma^i z)) \Bigg] \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \ln \Bigg[\sum_{x \in D_n(y)} \prod_{i=0}^{n-1} \exp\big(f(\sigma^i x)\big) \Bigg] = \mathcal{T}(y). \end{aligned}$$

Suppose that there exist $y \in Y$ and $\epsilon > 0$ for which

$$\theta(y) = \mathcal{T}(y) - 2\epsilon.$$

Since f is uniformly continuous, there is $p \geq 0$ such that whenever $\rho(z,x) < 2^{-p}$ or, equivalently, $z_{[-p,p]} = x_{[-p,p]}$, then $|f(z) - f(x)| < \epsilon$. Fix $x \in X$ and $n > 2p$. For each $i = p, p+1, \cdots, n-p-1$,

$$\begin{aligned} F_n^i(x_0 x_1 \cdots x_{n-1}) &= \inf_{z \in [x_0 \cdots x_{n-1}]} \exp(f(\sigma^i z)) \\ &\geq \inf_{\substack{z \in X \\ \rho(\sigma^i z, \sigma^i x) < 2^{-p}}} \exp(f(\sigma^i z)) \geq \exp\big(f(\sigma^i x) - \epsilon\big). \end{aligned}$$

Since $F_n^i \geq 1$ for each i and $\exp(f) \leq M$, we have

$$\begin{aligned} s(x_0 x_1 \cdots x_{n-1}) &\geq \prod_{i=p}^{n-p-1} F_n^i(x_0 x_1 \cdots x_{n-1}) \geq \prod_{i=p}^{n-p-1} \exp\big(f(\sigma^i x) - \epsilon\big) \\ &\geq M^{-2p} e^{-(n-2p)\epsilon} \prod_{i=0}^{n-1} \exp(f(\sigma^i x)). \end{aligned}$$

It follows that

$$\begin{aligned} \theta(y) &\geq \limsup_{n \to \infty} \frac{1}{n} \ln \left[M^{-2p} e^{-(n-2p)\epsilon} \sum_{x \in D_n(y)} \prod_{i=0}^{n-1} \exp(f(\sigma^i x)) \right] \\ &= \limsup_{n \to \infty} \frac{1}{n} \ln \left[\sum_{x \in D_n(y)} \prod_{i=0}^{n-1} \exp(f(\sigma^i x)) \right] - \epsilon = \mathcal{T}(y) - \epsilon, \end{aligned}$$

which is a contradiction. Therefore $\mathcal{T}(y) = \theta(y)$.

Observe next that for each $y \in Y$,

$$\begin{aligned} \theta(y) &= \limsup_{n \to \infty} \frac{1}{n} \ln \left[\sum_{x \in D_n(y)} s(x_0 x_1 \cdots x_{n-1}) \right] \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \ln \left[\sum_{\substack{\pi(x_0 \cdots x_{n-1}) \\ = y_0 \cdots y_{n-1}}} s(x_0 x_1 \cdots x_{n-1}) \right] = \Psi(y), \end{aligned}$$

since the summation for $\Psi(y)$ is taken over the larger set than the one for $\theta(y)$. As we have seen before, it is possible to have $\theta(y) < \Psi(y)$ for some $y \in Y$. We will prove that if $y \in Y$ is periodic, then $\theta(y) = \Psi(y)$, or, equivalently $\mathcal{T}(y) = \Psi(y)$.

Fix $y \in Y$ and for each $n \geq 1$, let

$$\tau_n(y) = \sum_{x \in D_n(y)} s(x_0 x_1 \cdots x_{n-1}).$$

Assuming that $D_n(y) \subset D_{n+1}(y)$ for each $n \geq 1$ (without loss of generality), we have

$$\tau_n(y) = \sum_{x \in D_n(y)} \prod_{i=0}^{n-1} \inf_{z \in x_{[0,n-1]}} \exp(f(\sigma^i z)) \leq \sum_{x \in D_n(y)} \prod_{i=0}^{n-1} \inf_{z \in x_{[0,n]}} \exp(f(\sigma^i z))$$
$$\leq \sum_{x \in D_{n+1}(y)} \prod_{i=0}^{n} \inf_{z \in x_{[0,n]}} \exp(f(\sigma^i z)) = \tau_{n+1}(y).$$

Hence $\tau_n(y)$ increases as n increases. An easy computation shows that given an increasing sequence $\{a_n\}$, for any $q \geq 1$, $\limsup_{n\to\infty} a_n/n = \limsup_{n\to\infty} a_{nq}/(nq)$. Letting $a_n = \ln \tau_n(y)$ proves that for any $q \geq 1$,

$$\theta(y) = \limsup_{n\to\infty} \frac{1}{nq} \ln \Bigg[\sum_{x \in D_{nq}(y)} s(x_0 x_1 \cdots x_{nq-1}) \Bigg]. \tag{5}$$

Let $y \in P_q(Y)$, $q \geq 1$. We claim that

$$\begin{aligned} \Psi(y) &= \limsup_{n\to\infty} \frac{1}{nq} \ln S(y_0 y_1 \cdots y_{nq-1}) \\ &= \limsup_{n\to\infty} \frac{1}{nq} \ln \Bigg[\sum_{\substack{\pi(x_0 \cdots x_{nq-1}) \\ = y_0 \cdots y_{nq-1}}} s(x_0 x_1 \cdots x_{nq-1}) \Bigg]. \end{aligned} \tag{6}$$

Then, using (5) and (6), we can proceed as in Lemma 2 to show that $\Psi(y) \leq \theta(y)$ and therefore $\Psi(y) = \mathcal{T}(y)$.

To verify (6), let $\epsilon > 0$ be given and (using uniform continuity of f) choose $p \geq 0$ so that whenever $\rho(z,x) < 2^{-p}$ or $z_{[-p,p]} = x_{[-p,p]}$, then $|f(z) - f(x)| < \epsilon$. Given an integer $m \geq 2p + q$, there is $n \in \mathbb{N}$ such that $2p < nq \leq m < (n+1)q$. Put $k = m - nq \geq 0$ and $v = y_0 y_1 \cdots y_{nq-1}$. Then

$$\begin{aligned} S(y_0 y_1 \ldots y_{m-1}) &= \sum_{\pi(u) = y_0 \cdots y_{m-1}} \prod_{i=0}^{m-1} \inf_{z \in [u]} \exp(f(\sigma^i z)) \\ &\leq \sum_{\pi(u) = v} \sum_{\substack{w \in \mathcal{B}_k(X) \\ uw \in \mathcal{B}(X)}} \prod_{i=0}^{m-1} \inf_{z \in [uw]} \exp(f(\sigma^i z)) \\ &\leq \sum_{\pi(u) = v} \sum_{\substack{w \in \mathcal{B}_k(X) \\ uw \in \mathcal{B}(X)}} M^{k+2p} \prod_{i=p}^{nq-p-1} \inf_{z \in [uw]} \exp(f(\sigma^i z)). \end{aligned} \tag{7}$$

Let $u \in \mathcal{B}_{nq}(X)$ and $w \in \mathcal{B}_k(X)$ be any blocks such that $uw \in \mathcal{B}(X)$. Let $p \le i \le nq-p-1$. Note that if $z \in [uw]$ and $\bar{z} \in [u]$, then $\rho(\sigma^i z, \sigma^i \bar{z}) < 2^{-p}$ so that $|f(\sigma^i z) - f(\sigma^i \bar{z})| < \epsilon$. Thus

$$\inf_{z\in[uw]} \exp(f(\sigma^i z)) \le \inf_{z\in[u]} \exp\big(f(\sigma^i z) + \epsilon\big).$$

From this inequality and (7), we have

$$\begin{aligned} S(y_0 y_1 \dots y_{m-1}) &\le M^{k+2p} \sum_{\pi(u)=v} \sum_{\substack{w\in\mathcal{B}_k(X)\\ uw\in\mathcal{B}(X)}} \prod_{i=p}^{nq-p-1} \inf_{z\in[u]} \exp\big(f(\sigma^i z)+\epsilon\big) \\ &\le M^{k+2p} \cdot \big|\mathcal{B}_k(X)\big| \sum_{\pi(u)=v} \prod_{i=0}^{nq-1} \inf_{z\in[u]} \exp\big(f(\sigma^i z)+\epsilon\big) \\ &= M^{k+2p} \cdot \big|\mathcal{B}_k(X)\big| \cdot e^{nq\epsilon} \cdot S(y_0 y_1 \dots y_{nq-1}). \end{aligned}$$

Since $k < q$ and p, q are fixed, it follows that

$$\begin{aligned} \Psi(y) &= \limsup_{m\to\infty} \frac{1}{m} \ln S(y_0 \cdots y_{m-1}) \\ &\le \limsup_{n\to\infty} \frac{1}{nq} \ln S(y_0 y_1 \dots y_{nq-1}) + \epsilon. \end{aligned}$$

Taking ϵ arbitrarily small completes the proof of the claim.

Next, we will show that $\widetilde{\Psi}_f(y) = \mathcal{T}(y)$ for $y \in P(Y)$. It is straightforward to check that $\mathcal{T}(y) \le \widetilde{\Psi}_f(y)$ for all $y \in Y$. Set $\widetilde{s} = \widetilde{s}_f$, $\widetilde{S} = \widetilde{S}_f$, and $\widetilde{\Psi} = \widetilde{\Psi}_f$. Define $\widetilde{\theta} : Y \to \mathbb{R}$ by

$$\widetilde{\theta}(y) = \limsup_{n\to\infty} \frac{1}{n} \ln \Bigg[\sum_{x\in D_n(y)} \widetilde{s}(x_0 x_1 \cdots x_{n-1}) \Bigg].$$

Note that $\mathcal{T}(y) \le \widetilde{\theta}(y)$. Suppose there exist $y \in Y$ and $\epsilon > 0$ for which

$$\widetilde{\theta}(y) = \mathcal{T}(y) + 2\epsilon.$$

Similarly to the foregoing, there is $p \ge 0$ such that whenever $\rho(z, x) < 2^{-p}$ or equivalently, $z_{[-p,p]} = x_{[-p,p]}$, then $|f(z) - f(x)| < \epsilon$. Fix $x \in X$ and $n > 2p$. If $p \le r \le n-p-1$, then

$$\begin{aligned} \widetilde{F}_n^r(x_0 x_1 \cdots x_{n-1}) &= \sup_{\sigma^{-r} z\in[x_0\cdots x_{n-1}]} \exp(f(z)) \\ &\le \sup_{\rho(z,\sigma^r x)<2^{-p}} \exp(f(z)) \le \exp(f(\sigma^r x) + \epsilon). \end{aligned}$$

Thus

$$
\begin{aligned}
&\widetilde{s}(x_0x_1\cdots x_{n-1})\\
&\quad=\prod_{r=0}^{p-1}\widetilde{F}_n^r(x_0\cdots x_{n-1})\prod_{r=p}^{n-p-1}\widetilde{F}_n^r(x_0\cdots x_{n-1})\prod_{r=n-p}^{n-1}\widetilde{F}_n^r(x_0\cdots x_{n-1})\\
&\quad\le M^{2p}\prod_{r=p}^{n-p-1}\widetilde{F}_n^r(x_0\cdots x_{n-1})\le M^{2p}\prod_{r=p}^{n-p-1}\exp(f(\sigma^r x)+\epsilon)\\
&\quad\le M^{2p}e^{(n-2p)\epsilon}\exp\Big(\sum_{r=p}^{n-p-1}f(\sigma^r x)\Big)\le M^{2p}e^{(n-2p)\epsilon}\exp\Big(\sum_{r=0}^{n-1}f(\sigma^r x)\Big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\widetilde{\theta}(y)&\le\limsup_{n\to\infty}\frac{1}{n}\ln\Big[M^{2p}e^{(n-2p)\epsilon}\sum_{x\in D_n(y)}\exp\Big(\sum_{r=0}^{n-1}f(\sigma^r x)\Big)\Big]\\
&=\epsilon+\limsup_{n\to\infty}\frac{1}{n}\ln\Big[\sum_{x\in D_n(y)}\exp\Big(\sum_{r=0}^{n-1}f(\sigma^r x)\Big)\Big]=\epsilon+\mathcal{T}(y),
\end{aligned}
$$

which is a contradiction. Therefore $\mathcal{T}(y)=\widetilde{\theta}(y)$.

Let $y\in P_q(Y)$, $q\ge 1$. It is not difficult to see that

$$
\mathcal{T}(y)=\limsup_{n\to\infty}\frac{1}{nq}\ln\Big[\sum_{x\in D_{nq}(y)}\widetilde{s}(x_0x_1\cdots x_{nq-1})\Big]=\widetilde{\theta}(y)
$$

and

$$
\widetilde{\Psi}(y)=\limsup_{n\to\infty}\frac{1}{nq}\ln\widetilde{S}(y_0\cdots y_{nq-1}).
$$

We can proceed again as in Lemma 2 to show that $\widetilde{\Psi}(y)=\mathcal{T}(y)$. The remainder of the proof is the same as before. □

Recall that according to Theorem 4.6 of [6], if for each $n=1,2,\cdots$ and $y\in Y$ we denote by $D_n(y)$ a set consisting of exactly one point from each nonempty set $[x_0\cdots x_{n-1}]\cap\pi^{-1}(y)$, then for each $f\in C(Y)$,

$$
P(\pi,f)(y)=\limsup_{n\to\infty}\frac{1}{n}\ln\Big[\sum_{x\in D_n(y)}\exp\Big(\sum_{i=0}^{n-1}f(\sigma^i x)\Big)\Big].
$$

From Theorem 2 it now follows that we will obtain the value of $P(\pi,f)(y)$ a.e. with respect to every invariant measure on Y if we delete from the definition of $D_n(y)$ the requirement that $x\in\pi^{-1}(y)$:

Corollary. For each $n = 1, 2, \cdots$ and $y \in Y$ denote by $E_n(y)$ a set consisting of exactly one point from each nonempty cylinder $[x_0 \cdots x_{n-1}] \subset \pi^{-1}[y_0 \cdots y_{n-1}]$. Then for each $f \in C(Y)$,

$$P(\pi, f)(y) = \limsup_{n\to\infty} \frac{1}{n} \ln \left[\sum_{x\in E_n(y)} \exp\left(\sum_{i=0}^{n-1} f(\sigma^i x) \right) \right]$$

a.e. with respect to every invariant measure on Y.

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